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CONTINUOUS SELECTIONS OF THE INVERSE NUMERICAL RANGE MAP

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Abstract. For a complex n-by-n matrix A, the numerical range $F(A)$ is the range of the map $f_A(x) = x^*Ax$ acting on the unit sphere in $\mathbb{C}^n$. We ask whether the multivalued inverse numerical range map f_A^{-1} has a continuous single-valued selection defined on all or part of $F(A)$. We show that for a large class of matrices, f_A^{-1} does have a continuous selection on $F(A)$. For other matrices, f_A^{-1} has a continuous selection defined everywhere on $F(A)$ except in the vicinity of a finite number of exceptional points on the boundary of $F(A)$.

1. Introduction

The *numerical range* (also known as the *field of values*) $F(A)$ of a matrix $A \in M_n(\mathbb{C})$ is the image of the complex unit sphere $\mathbb{C}S^n = \{x \in \mathbb{C}^n : x^*x = 1\}$ under the map $f_A(x) = x^*Ax$. Since the map $f_A(x)$ is continuous, the numerical range is always a compact, connected subset of $\mathbb{C}$. It contains the eigenvalues of A and is convex [5]. These properties make the numerical range a useful tool in applications and within linear algebra.

Recently, several papers have studied the pre-images of a complex number $z \in F(A)$ under the map f_A. For any unimodular constant $\omega \in \mathbb{C}$ and $x \in \mathbb{C}S^n$, $f_A(\omega x) = f_A(x)$. It follows that the pre-images of f_A will always be trivially multivalued. In fact, $f_A^{-1}(z)$ contains a set of n linearly independent vectors for every $z \in \operatorname{int} F(A)$ [2, Theorem 1]. Algorithms for computing at least one element of $f_A^{-1}(z)$ are presented in [2, 3, 13, 17].

As a multivalued map, there are several possible notions of continuity that could apply to the inverse numerical range map f_A^{-1}. In [4] the following definitions were introduced. Let g be a multivalued mapping from a metric space (X, d_X) to a metric space (Y, d_Y). We say that g is *weakly continuous* at $x \in X$ if for all sequences $x_k \to x$ in X, there exists $y \in g(x)$ and a sequence $y_k \in g(x_k)$ such that $y_k \to y$. If such sequences exist for all $y \in g(x)$, then g is *strongly continuous* at x. Alternatively, f_A^{-1} is weakly (strongly) continuous at $z \in F(A)$ if the direct mapping f_A is open with respect to the relative topology on $F(A)$ at some (resp., all) pre-images $x \in f_A^{-1}(z)$. In [4], it was shown that the inverse field of values map is strongly continuous on the interior of $F(A)$, and that strong continuity can only fail at so-called round points of the boundary. Necessary and sufficient conditions for weak and strong continuity of f_A^{-1} are given in [11]. In particular, strong (and

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therefore weak) continuity can only fail at finitely many points on the boundary [11, Corollary 2.3].

Related notions of continuity for multivalued maps include the notions of upper and lower semi-continuity [14]. As the inverse of a continuous single-valued function, f_A^{-1} is automatically upper semi-continuous. In our terminology, strong continuity is equivalent to lower semi-continuity.

Given any multivalued function $g : X \rightrightarrows Y$, we may also ask whether there exists a continuous single valued function $h : X \to Y$ such that $h(x) \in g(x)$ for all $x \in X$. Such a function is called a *continuous selection* of g. There are several general theorems due to Michael [14, 15] concerning whether upper and lower semi-continuous multivalued functions admit a continuous single-valued selection. These theorems require additional convexity or connectedness assumptions in order to apply. In the case of the map f_A^{-1} the convexity assumptions do not apply, and the connectedness assumptions are difficult to verify. By the definition of weak continuity, if G is a relatively open subset of $F(A)$ containing a point where f_A^{-1} is not weakly continuous, then it is not possible to define a continuous selection of f_A^{-1} on G.

The main result of this paper is the following theorem.

Theorem 1. *Let $A \in M_n(\mathbb{C})$. If f_A^{-1} has no weak continuity failures on $F(A)$, then there is a continuous selection of f_A^{-1} on all of $F(A)$. If f_A^{-1} has weak continuity failures at $w_1, \dots, w_k \in \partial F(A)$, then for any open set G containing $\{w_1, \dots, w_k\}$ there is a continuous selection of f_A^{-1} on $F(A)\backslash G$.*

Note that f_A^{-1} is weakly continuous everywhere on $F(A)$ when A is normal and when $n \le 3$ [4, Corollaries 5 and 6, and Theorem 11]. For such matrices it follows that f_A^{-1} has a continuous selection defined on all of $F(A)$. More generally, the set of matrices $A \in M_n(\mathbb{C})$ for which f_A^{-1} is strongly (and therefore also weakly) continuous on all of $F(A)$ is generic [11, Proposition 2.4].

2. Preliminaries

When considering the map f_A, it is natural to identify vectors that are scalar multiples in $\mathbb{C}S^n$. Under this identification, $\mathbb{C}S^n$ becomes the complex projective space $\mathbb{C}P^{n-1}$. With the inclusion map $q : \mathbb{C}S^n \to \mathbb{C}P^{n-1}$, there is a unique map $\hat{f}_A : \mathbb{C}P^{n-1} \to F(A)$ that makes the diagram below commute.

$$\begin{tikzcd}
\mathbb{C}S^n \arrow[r, "f"] \arrow[d, "q"'] & F(A) \\
\mathbb{C}P^{n-1} \arrow[ur, dashed, "\hat{f}"'] &
\end{tikzcd}$$

Importantly, even the map $\hat{f}_A$ has multivalued pre-images since $f_A^{-1}(z)$ contains n linearly independent vectors in $\mathbb{C}S^n$ when z is in the interior of $F(A)$ [2, Theorem 1]. On the boundary of $F(A)$, $\hat{f}_A^{-1}$ may be single or multivalued [12].

The complex projective space $\mathbb{C}P^{n-1}$ is homeomorphic to the set $\{xx^* : x \in \mathbb{C}S^n\}$ via the map $\varphi : [x] \to xx^*/x^*x$. It will be convenient to use $\{xx^* : x \in \mathbb{C}S^n\}$ as a representation for $\mathbb{C}P^{n-1}$, so when we write $\mathbb{C}P^{n-1}$, we mean the set $\{xx^* : x \in \mathbb{C}S^n\}$. Thus, $\mathbb{C}P^{n-1} \subset H_n(\mathbb{C})$, where $H_n(\mathbb{C})$ denotes the set of n-by-n complex Hermitian matrices. Note that $\hat{f}_A(xx^*) = \operatorname{tr}(Axx^*)$, so $\hat{f}_A$ is the restriction to $\mathbb{C}P^{n-1}$ of a real linear map from the real vector space $H_n(\mathbb{C})$ to $\mathbb{C}$.

In order to present a continuous selection of f_A^{-1} on $V \subseteq F(A)$, it is sufficient to find a subset $U \subset \mathbb{C}S^n$ such that f_A is a bijection from U to V. It follows immediately from the compactness of $\mathbb{C}S^n$ that f_A^{-1} is a continuous function from V to U, so by restricting the range to V, we obtain a continuous selection. We may also find a subset $W \subset \mathbb{C}P^{n-1}$ on which $\hat{f}_A$ is a bijection onto V, in order to arrive at a continuous selection of $\hat{f}_A^{-1}$ on V. Our approach to finding sets on which f_A (or $\hat{f}_A$) is a bijection is to parametrize a subset $U \subset \mathbb{C}S^n$ ($W \subset \mathbb{C}P^{n-1}$) and show that this parametrization composed with f_A (respectively, $\hat{f}_A$) is a parametrization of the corresponding $V \subset F(A)$.

For any matrix $A \in M_n(\mathbb{C})$, recall the real and imaginary parts of A, $\operatorname{Re}(A) = (A + A^*)/2$ and $\operatorname{Im}(A) = (A - A^*)/2i$. Given $A \in M_n(\mathbb{C})$, note that $F(e^{-i\theta}A) = e^{-i\theta}F(A)$ for any $\theta \in [0, 2\pi)$. Furthermore, $e^{-i\theta}A = \operatorname{Re}(e^{-i\theta}A) + i\operatorname{Im}(e^{-i\theta}A)$. The left and right-most points on the rotated numerical range $F(e^{-i\theta}A)$ correspond to the maximal and minimal eigenvalues of the matrix $\operatorname{Re}(e^{-i\theta}A)$. The map $\theta \mapsto \operatorname{Re}(e^{-i\theta}A)$ is an analytic self-adjoint matrix valued function. By a theorem of Rellich [1, Corollary 2, section 3.5.5], there is a family of n functions $x_1(\theta), \ldots, x_n(\theta)$ that are analytic in θ on $[0, 2\pi]$ and form an orthonormal basis of eigenvectors for $\operatorname{Re}(e^{-i\theta}A)$. For each $j \in \{1, \ldots, n\}$ let $\lambda_j(\theta)$ denote the eigenvalue of $\operatorname{Re}(e^{-i\theta}A)$ corresponding to $x_j(\theta)$. Of course, the eigenvalue-functions $\lambda_j(\theta)$ are also analytic in θ since $\lambda_j = x_j^*\operatorname{Re}(e^{-i\theta}A)x_j$.

For each eigenvector-function $x_j(\theta)$ there is an associated *critical curve*, defined by $z_j(\theta) = f_A(x_j(\theta))$. The images of these critical curves are contained in the numerical range $F(A)$. Furthermore, $F(A)$ is precisely the convex hull of the critical curves. Using the fact that $\frac{d}{d\theta}\operatorname{Re}(e^{-i\theta}A) = i\operatorname{Im}(e^{-i\theta}A)$, we can derive the following alternative formula for the critical curves,

$$z_j(\theta) = e^{i\theta}(\lambda_j(\theta) + i\lambda_j'(\theta)). \tag{1}$$

The relationship between a numerical range and its critical curves was first described by Kippenhahn [10]. See also [8] and [7, Section 5] for more details.

From (1), it is clear that the eigenvalue-functions of $\operatorname{Re}(e^{-i\theta}A)$ determine the shape of the numerical range. Since the eigenvalue-functions λ_j are analytic, any two of them may only coincide at finitely many values of θ unless they are identical. Thus, for all but finitely many angles $\theta \in [0, 2\pi)$, $\operatorname{Re}(e^{-i\theta}A)$ has $m \le n$ distinct eigenvalues. We will call θ an *exceptional argument* if two or more distinct eigenvalue-functions coincide at θ. The corresponding points $z_j(\theta)$ given by (1) are *exceptional points.*

At an exceptional argument θ_0, there will be at least two distinct eigenvalue-functions achieving the same value. Since both eigenvalue-functions are analytic, both functions have Taylor series expansions about θ_0, which must differ. We say the the eigenvalue-functions *split at degree k* if the first coefficient where the Taylor series differ is in the degree k term. Note that flat portions of the boundary of $F(A)$ occur at exceptional arguments where the maximal eigenvalue-functions split

at degree one. This follows immediately from (1) and the fact that $F(A)$ is the convex hull of the critical curves. A corner point of $F(A)$ is a point contained at the intersection of two flat portions. A boundary point that is not a corner point and is not contained in the relative interior of a flat portion will be called a *round point*. If a round point is not an endpoint of a flat portion, then it is *fully round*.

The following theorem makes the relationship between the eigenvalue-functions of $\mathrm{Re}\,(e^{-i\theta}A)$ and continuity failures of f_A^{-1} clear.

Theorem 2 (Theorem 2.1, [11])**.** *Let $A \in M_n(\mathbb{C})$ and $z = z_j(\theta_0) \in \partial F(A)$.*

1. *f_A^{-1} is strongly continuous at z if and only if z is in the relative interior of a flat portion of the boundary or the eigenvalue-functions corresponding to z at θ_0 do not split.*
2. *f_A^{-1} is weakly continuous at z if and only if z is in a flat portion of the boundary or the eigenvalue-functions corresponding to z at θ_0 do not split at odd powers. So weak continuity fails if and only if z is a fully round boundary point and the eigenvalue-functions corresponding to z at θ_0 split at an odd power.*

An immediate consequence of Theorem 2 is that strong (and therefore weak) continuity of f_A^{-1} can only fail at exceptional points on the boundary of the numerical range. In particular, there are at most finitely many exceptional points where weak continuity can fail.

3. Continuous Selections on the Boundary

Lemma 1. *For any analytic curve $\Gamma \subseteq \partial F(A)$, there is an analytic path $x : [0,1] \to \mathbb{C}S^n$ such that $f_A(x(t))$ parametrizes Γ. If $F(A)$ has nonempty interior and Γ is the whole boundary of $F(A)$, then x may be chosen to be periodic on $[0,1]$. The intersection of f_A^{-1} with the range of $x(t)$ is a continuous selection of f_A^{-1} on Γ.*

Proof. The analytic curve Γ is either contained in one of the critical curves of $F(A)$, or it is contained in a flat portion of the boundary. In the former case, there is one eigenvalue-function $\lambda(\theta)$ corresponding to the maximal eigenvalue of $\mathrm{Re}\,(e^{-i\theta}A) = \cos(\theta)H + \sin(\theta)K$ such that Γ is parametrized by (1) for θ in some closed interval $I \subseteq [0, 2\pi]$. Let $P(\theta)$ denote the corresponding spectral projection, which is also an analytic function of θ ([9, Theorem II.6.1]). The map $\varphi : (v,\theta) \mapsto v - P(\theta)v$ has differentials with real rank at most $2n-1$, so by Sard's theorem [16] the range of φ must have measure zero in $\mathbb{C}^n$. Choose a $w \in \mathbb{C}^n$ that is not in the range of φ. Note that $P(\theta)w \neq 0$ for all $\theta \in I$, otherwise $w - P(\theta)w = w$ which would contradict our assumption that w is not in the range of φ. Now let $x(\theta) = P(\theta)w/||P(\theta)w||$. By construction, $x(\theta)$ is a unit eigenvector of $\mathrm{Re}\,(e^{-i\theta}A)$ corresponding to the maximal eigenvalue $\lambda(\theta)$, and therefore $f_A(x(\theta))$ parametrizes Γ. We can make a simple affine linear change of variables to replace $x(\theta)$ defined on I with $x(t)$ defined on $[0,1]$. If Γ is a closed loop, then the corresponding spectral projection $P(\theta)$ is periodic on $[0, 2\pi]$. It follows that our construction of $x(t)$ is periodic.

If Γ is a subset of a flat portion of the boundary, then let $x, y \in \mathbb{C}S^n$ be preimages under f_A of the two endpoints of the flat portion. There is an angle $\theta \in [0, 2\pi)$ such that x, y are both eigenvectors of $\mathrm{Re}\,(e^{-i\theta}A)$ corresponding to the maximal eigenvalue. If A_2 denotes the compression of A corresponding to the 2-by-2 subspace $\mathrm{Span}\,\{x, y\}$, then x, y are the eigenvectors of $\mathrm{Im}\,(e^{-i\theta}A_2)$ corresponding to the

maximal and minimal eigenvalues. This implies that x and y are orthogonal, and direct computation shows that the flat portion can be parametrized by $f_A(x\cos\omega + y\sin\omega)$ for $\omega \in [0, \pi/2]$. If we denote $x(\omega) = x\cos\omega + y\sin\omega$, we can make a simple change of variables so that $x(t)$ has domain $[0,1]$. □

If f_A^{-1} has weak continuity failures on $\partial F(A)$, then it will not be possible to choose a continuous selection on the whole boundary. If there are no weak continuity failures on the boundary, then it is possible to find a continuous selection as the following lemma proves.

Lemma 2. *Suppose that f_A^{-1} is weakly continuous on all of $F(A)$. Then there is a continuous selection of f_A^{-1} on $\partial F(A)$.*

Proof. In the case where $F(A)$ has no flat portions, the boundary is given by a single critical curve, and Lemma 1 immediately implies that there is an analytic, periodic function $x : [0,1] \to \mathbb{C}S^n$ such that $f_A(x(t))$ parametrizes the boundary. The intersection of f_A^{-1} with the range of x is a continuous selection.

If there are flat portions, then apply Lemma 1 to chose a continuous selection of f_A^{-1} on each curved analytic portion of the boundary. Since there are no weak continuity failures, the maximal eigenvalue-functions that define the boundary of $F(A)$ can only cross at degree one splitting points. Therefore the boundary of $F(A)$ is defined by alternating analytic curves and flat portions. If $F(A)$ has corner points, one or more of the analytic curves may be single points, but that is not a concern. For a given flat portion, let $x, y \in \mathbb{C}S^n$ be the pre-images of the end points as determined by the continuous selections on the curved portions of $\partial F(A)$. Using Lemma 1, we choose an $x : [0,1] \to \mathbb{C}S^n$ such that $f_A(x(t))$ parametrizes the flat portion of the boundary. From the proof of Lemma 1, it is clear that we may choose $x(t)$ such that $x(0) = x$ and $x(1) = y$. Then the map $f_A(x(t)) \mapsto x(t)$ continuously extends our selection of f_A^{-1} to include the flat portion. □

4. Constructing the Selection

In order to construct a continuous selection of f_A^{-1} on the interior of $F(A)$, it will be convenient to have an alternative formula for a selection of f_A^{-1} on $\partial F(A)$. Let $A \in M_n(\mathbb{C})$ have a numerical range $F(A)$ with non-empty interior. Suppose that the critical curves corresponding to the maximal eigenvalues of $\mathrm{Re}\,(e^{-i\theta}A)$ cross at the exceptional arguments $\theta_1 < \theta_2 < \ldots < \theta_m$ in $[0, 2\pi)$. By rotation, we may assume without loss of generality that $\theta_1 > 0$. Fix $x_0 \in f_A^{-1}(z_0)$ where z_0 is the point in $F(A)$ with maximal real part.

On each interval (θ_k, θ_{k+1}), the spectral projection $P(\theta)$ corresponding to the maximal eigenvalue of $\mathrm{Re}\,(e^{-i\theta}A)$ is an analytic function of θ. The projection valued function $P(\theta)$ on (θ_k, θ_{k+1}) extends to an analytic function on $\mathbb{R}$ [9, Theorem II.6.1], which we will denote by $P_k(\theta)$. For all values of θ, $P_k(\theta)$ is a projection into an eigenspace of $\mathrm{Re}\,(e^{-i\theta}A)$, however the corresponding eigenvalue may not be maximal outside (θ_k, θ_{k+1}).

The expression $P_k(\theta)x_0$ is analytic, as is $||P_k(\theta)x_0||$. If $||P_k(\theta)x_0||$ is not identically zero on $[\theta_k, \theta_{k+1}]$, then it must be nonzero for all but finitely many θ in the interval. In this case, we claim that there is a piecewise function $\alpha : [\theta_k, \theta_{k+1}] \to$

$\{-1,1\}$ such that

$$x_k(\theta) = \alpha(\theta)\frac{P_k(\theta)x_0}{||P_k(\theta)x_0||} \tag{2}$$

has only removable discontinuities, and so we may extend x_k to a continuous function on $[\theta_k, \theta_{k+1}]$. The following lemma proves this claim.

Lemma 3. *Let I be an interval in $\mathbb{R}$, and suppose that $x : I \to \mathbb{C}^n$ is analytic. There is a continuous function $y : I \to \mathbb{C}^n$ such that $y(t) = \pm x(t)/||x(t)||$ for all $t \in I$ where $x(t) \neq 0$.*

Proof. Since $x(t)$ is analytic, so is $x(t)/||x(t)||$, except at points where $x(t) = 0$. Let $t_1, \ldots, t_m$ denote these zeros. Near each zero t_j, $x(t)$ has a Taylor series expansion:

$$x(t) = a_1(t - t_j) + a_2(t - t_j)^2 + \ldots.$$

Let k_j denote the degree of the first nonzero vector coefficient in the series above. Note that

$$\lim_{t \to t_j^-} \frac{x(t)}{||x(t)||} = \frac{a_{k_j}}{||a_{k_j}||}(-1)^{k_j},$$

and

$$\lim_{t \to t_j^+} \frac{x(t)}{||x(t)||} = \frac{a_{k_j}}{||a_{k_j}||}.$$

Let $y(t) = c_j \frac{x(t)}{||x(t)||}$ on each open interval between adjacent zeros t_j and t_{j+1}, where each c_j is either 1 or -1. By choosing the c_j constants sequentially, we can ensure that the discontinuities in $y(t)$ at each t_j are removable, and therefore $y(t)$ can be extended to a continuous function on I. □

If $P_k(\theta)x_0 = 0$ identically on $[\theta_k, \theta_{k+1}]$, then we choose w as in the proof of Lemma 1 such that $P_k(\theta)w \neq 0$ for all $\theta \in [\theta_k, \theta_{k+1}]$. In this case we let

$$x_k(\theta) = \frac{P_k(\theta)w}{||P_k(\theta)w||} \tag{3}$$

In both (2) and (3), we note that $x_0^* x_k(\theta) \in \mathbb{R}$ for all $\theta \in [\theta_k, \theta_{k+1}]$.

If the maximal eigenvalue-function $\lambda(\theta)$ of $\mathrm{Re}\,(e^{-i\theta}A)$ splits at even degree at θ_k, then the spectral projections $P_k(\theta)$ and $P_{k-1}(\theta)$ are identical as are the functions $x_k(\theta)$ and $x_{k-1}(\theta)$ (see the proof of Theorem 2.1 in [11]). If $\lambda(\theta)$ splits at degree one then there is a flat portion of the boundary corresponding to the argument θ_k. On the flat portion, we define the function

$$y_k(\omega) = \cos(\omega)x_{k-1}(\theta_k) + \sin(\omega)x_k(\theta_k), \tag{4}$$

and we note that f_A is a bijection from the curve $\{y_k(\omega) : \omega \in [0, \pi/2]\}$ in $\mathbb{C}S^n$ onto the flat portion of the boundary corresponding to the argument θ_k. By traversing the curves x_k and y_k in order and parametrizing the resulting curve with domain $[0, 1]$, we obtain a path $y(t)$ in $\mathbb{C}S^n$ such that the image of $f_A(y(t))$ is the boundary of $F(A)$, $x_0^* y(t) \in \mathbb{R}$ for all $t \in [0, 1]$, and $y(t)$ is continuous except at values of t where $y(t)$ corresponds to an exceptional argument θ_k where $\lambda(\theta)$ splits at odd degree greater than 1. For these $y(t)$, $f_A(y(t))$ is a point on $\partial F(A)$ where weak continuity of f_A^{-1} fails by Theorem 2. If there are no weak continuity failures on $\partial F(A)$, then $y(t)$ is continuous on $[0, 1]$, although $x_0 = y(0)$ and $y(1)$ may differ by a constant. We will refer to functions $y(t)$ constructed in this manner as *canonical selections of $\partial F(A)$*.

Lemma 4. *Let A be a non-normal 2-by-2 matrix with complex entries, and suppose that $x, y \in \mathbb{C}S^2$, $x^*y \neq 0$, and $f_A(x)$ an $f_A(y)$ are distinct points in $\partial F(A)$. Let*

$$h(\lambda) = \frac{\lambda x + (1-\lambda)\left(y^*x + i\beta\sqrt{1-|x^*y|^2}\right)y + \frac{\sqrt{2}}{2}Cv}{\sqrt{C+1}}, \tag{5}$$

where $C = 2\sqrt{\lambda(1-\lambda)(1-|x^*y|)}$, $\beta = \pm\frac{y^*x}{|x^*y|}$, and $v = \frac{\sqrt{2}}{2}\left(x + i\beta\frac{y-(x^*y)x}{||y-(x^*y)x||}\right)$. Then

$$f_A(h(\lambda)) = \lambda f_A(x) + (1-\lambda)f_A(y).$$

Proof. By the well known Elliptical Range Theorem, $F(A)$ is a convex ellipse. An elegant proof of that theorem can be found in [5]. The main observation of [5] is that $\mathbb{C}P^1$ is a 2-sphere of radius $\frac{\sqrt{2}}{2}$ centered at $\frac{1}{2}I_2$ in the affine subspace of $H_2(\mathbb{C})$ consisting of matrices with trace one. Therefore $F(A)$, which is the image of $\mathbb{C}P^1$ under the linear transformation $\hat{f}_A$, must be a convex ellipse. Since A is not normal, the ellipse is not degenerate (see e.g., [6]).

The following matrices form a basis for the set of trace zero matrices in $H_2(\mathbb{C})$.

$$X_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, X_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Consider the linear map $\psi : H_2(\mathbb{C}) \to \mathbb{R}^3$ defined by

$$\psi(Y) = \begin{bmatrix} \langle Y, X_1\rangle \\ \langle Y, X_2\rangle \\ \langle Y, X_3\rangle \end{bmatrix}.$$

The image of $\mathbb{C}P^1$ under ψ is precisely the unit sphere S in $\mathbb{R}^3$. In fact, if we restrict the domain of ψ to $\mathbb{C}P^1$, then $\psi : \mathbb{C}P^1 \to S$ is a bijection and for any $r = (r_1, r_2, r_3) \in S$, $\psi^{-1}(r) = \frac{1}{2}(r_1X_1 + r_2X_2 + r_3X_3) + \frac{1}{2}I_2$. Observe the following relations:

- For $x, y \in \mathbb{C}S^n$, $\langle xx^*, yy^*\rangle_{H_n(\mathbb{C})} = |x^*y|^2$.
- For $xx^*, yy^* \in \mathbb{C}P^1$, $\langle xx^*, yy^*\rangle_{H_n(\mathbb{C})} = \frac{1}{2}\langle\psi(xx^*), \psi(yy^*)\rangle_{\mathbb{R}^3} + \frac{1}{2}$.

Let $H = \mathrm{Re}\,(A)$ and $K = \mathrm{Im}\,(A)$ so that $\hat{f}_A(Y) = \mathrm{tr}\,(HY) + i\mathrm{tr}\,(KY)$. If we identify $\mathbb{C}$ with $\mathbb{R}^2$, then for any $u \in \mathbb{C}S^2$,

$$\hat{f}_A(uu^*) = B\psi(uu^*) + \tfrac{1}{2}\mathrm{tr}\,(A)$$

where

$$B = \tfrac{1}{2}\begin{bmatrix} \langle H, X_1\rangle & \langle H, X_2\rangle & \langle H, X_3\rangle \\ \langle K, X_1\rangle & \langle K, X_2\rangle & \langle K, X_3\rangle \end{bmatrix}.$$

Since xx^* and yy^* are both mapped to $\partial F(A)$ by $\hat{f}_A$, it follows that $\psi(xx^*)$ and $\psi(yy^*)$ are both on the great circle of S that is mapped to $\partial F(A)$ by the affine linear transformation $r \mapsto Br + \frac{1}{2}\mathrm{tr}\,(A)$. The two vectors in S orthogonal to this great circle are in the nullspace of B. Since $|x^*v| = \frac{\sqrt{2}}{2}$ and

$$y^*v = \frac{\sqrt{2}}{2}\left(y^*x + i\beta\frac{1-|x^*y|^2}{||y-(x^*y)x||}\right) = \frac{\sqrt{2}}{2}\left(y^*x + i\beta\sqrt{1-|x^*y|^2}\right), \tag{6}$$

so that $|y^*v| = \frac{\sqrt{2}}{2}$, it follows that $\psi(vv^*)$ is orthogonal to both $\psi(xx^*)$ and $\psi(yy^*)$. Since $x^*y \neq 0$, $\psi(xx^*)$ and $\psi(yy^*)$ are not antipodal points on S. Therefore $\psi(vv^*)$ must be one of the two vectors in S that are in the nullspace of B.

We now construct a point $s \in S$ such that $Bs + \frac{1}{2}\operatorname{tr} A = \lambda f_A(x) + (1-\lambda) f_A(y)$. The point $r = \lambda\psi(xx^*) + (1-\lambda)\psi(yy^*)$, has $Br + \frac{1}{2}\operatorname{tr} A = \lambda f_A(x) + (1-\lambda) f_A(y)$ by construction, but $r \notin S$. Note that

$$||r||^2 = \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\langle \psi(xx^*), \psi(yy^*)\rangle_{\mathbb{R}^3} =$$

$$= 2\lambda^2 - 2\lambda + 1 + 2\lambda(1-\lambda)(2|x^*y|^2 - 1).$$

Let $C^2 = 1 - ||r||^2$. Then

$$C^2 = 2\lambda - 2\lambda^2 - (2\lambda - 2\lambda^2)(2|x^*y|^2 - 1) =$$

$$= 4\lambda(1-\lambda)(1 - |x^*y|^2).$$

So $C = 2\sqrt{\lambda(1-\lambda)(1-|x^*y|^2)}$. Since $r \perp \psi(vv^*)$, it follows that $s = r + C\psi(vv^*) \in S$ has the desired properties. By applying the map ψ^{-1} to s, we see that there must be some $u \in \mathbb{C}S^2$ such that

$$uu^* = \psi^{-1}(s) = \lambda xx^* + (1-\lambda)yy^* + C(vv^* - \tfrac{1}{2}I_2) \in \mathbb{C}P^1,$$

and $f_A(u) = \hat{f}_A(uu^*) = \lambda f_A(x) + (1-\lambda) f_A(y)$. Given uu^* it is only possible to determine u up to multiplication by a unimodular constant. However, it is convenient to set

$$u = \frac{uu^*v}{||uu^*v||} = \frac{uu^*v}{|u^*v|} = \frac{uu^*v}{\sqrt{\langle uu^*, vv^*\rangle_{H_2(\mathbb{C})}}} = \frac{\sqrt{2}uu^*v}{\sqrt{C+1}} =$$

$$= \frac{\lambda x + \sqrt{2}(1-\lambda)(y^*v)y + \frac{\sqrt{2}}{2}Cv}{\sqrt{C+1}}.$$

By (6), $\sqrt{2}y^*v = y^*x + i\beta\sqrt{1-|x^*y|^2}$. Letting $h(\lambda) = u$ gives

$$h(\lambda) = \frac{\lambda x + (1-\lambda)\left(y^*x + i\beta\sqrt{1-|x^*y|^2}\right)y + \frac{\sqrt{2}}{2}Cv}{\sqrt{C+1}}.$$

□

For normal 2-by-2 matrices, the following result can be verified by direct computation.

Lemma 5. *Let $A \in M_2(\mathbb{C})$ be normal with two distinct eigenvalues, and suppose that $x, y \in \mathbb{C}S^2$ are eigenvectors corresponding to the two eigenvalues of A. Let*

$$h(\lambda) = \frac{\lambda x + (1-\lambda)iy + \frac{\sqrt{2}}{2}Cv}{\sqrt{C+1}}, \tag{7}$$

where $C = 2\sqrt{\lambda(1-\lambda)}$ and $v = \frac{\sqrt{2}}{2}(x+iy)$. Then

$$f_A(h(\lambda)) = \lambda f_A(x) + (1-\lambda) f_A(y).$$

Note that (7) is equivalent to (5) with $\beta = 1$. In fact, any unimodular constant β would have worked.

Theorem 3. *Let $A \in M_n(\mathbb{C})$ be a matrix such that $F(A)$ has no corner points and f_A^{-1} has no weak continuity failures on $F(A)$. There is a continuous selection $g : F(A) \to \mathbb{C}S^n$ of f_A^{-1}.*

Proof. Fix a fully round point $z_0 \in \partial F(A)$ such that the tangent line to z_0 in $F(A)$ is not parallel to any flat portions of the boundary of $F(A)$. By rotation, we may assume that z_0 is the right-most point in $F(A)$. Since there are no corner points, the boundary of $F(A)$ consists of alternating flat and round portions. On each round portion there is a selection of f_A^{-1} corresponding to one of the curves (2) or (3). On each flat portion there is a selection corresponding to (4). By traversing these curves in order as described at the beginning of the section, we may choose a continuous path $y : [0,1] \to \mathbb{C}S^n$ such that f_A is a bijection from $y([0,1))$ to $\partial F(A)$, $y(0) \in f_A^{-1}(z_0)$, $y(1)$ is a multiple of $y(0)$, and $y(0)^* y(t) \in \mathbb{R}$ for all $t \in (0,1)$.

Let $x_0 = y(0)$, $y = y(t)$. For any $z \in F(A)\backslash\{z_0\}$, there is a unique $\lambda \in [0,1)$ and $t \in (0,1)$ such that $z = \lambda z_0 + (1-\lambda) f_A(y(t))$. Furthermore, the values of λ and t vary continuously in z. We will prove that

$$g(z) = \frac{\lambda x_0 + (1-\lambda)\left(y^* x_0 + i\sqrt{1-|x_0^* y|^2}\right) y + \frac{\sqrt{2}}{2} C v}{\sqrt{C+1}}, \tag{8}$$

where $C = 2\sqrt{\lambda(1-\lambda)(1-|x_0^* y|^2)}$ and $v = \frac{\sqrt{2}}{2}\left(x_0 + i\frac{y-(x_0^* y)x_0}{||y-(x_0^* y)x_0||}\right)$, is a continuous selection of f_A^{-1} on $F(A)$. Note that g is continuous on $F(A)\backslash\{z_0\}$ by construction, and it is clear that if we set $g(z_0) = x_0$, then g extends continuously to all of $F(A)$. All that remains is to prove that g is a selection of f_A^{-1}.

Note that the tangent line to z_0 is vertical, and since by assumption there are no vertical flat portions of $\partial F(A)$, there is one $t \in (0,1)$ for which $f_A(y(t))$ is the unique leftmost point of $F(A)$. For all other $t \in (0,1)$, the tangent line to $f_A(y(t))$ in $F(A)$ is not vertical. Let A_2 be the 2-by-2 compression of A onto the $\operatorname{Span}\{x_0, y\}$. The numerical range of A_2 is a convex ellipse (possibly degenerate) and $z_0, f_A(y) \in F(A_2) \subseteq F(A)$. Recall that $F(A_2)$ is a line segment if and only if A_2 is normal. In that case, z_0 and $f_A(y)$ must be the endpoints of that line segment, since z_0 and $f_A(y)$ distinct points in the boundary of A. The endpoints of the line segment are the eigenvalues of the compression A_2, and thus x_0 and y are the eigenvectors corresponding to those eigenvalues. Applying Lemma 5 to the compression A_2 shows that g is a continuous selection of f_A^{-1} on the line segment from z_0 to $f_A(y(t))$.

When $F(A_2)$ is a non-degenerate ellipse, the tangent lines to z_0 and $f_A(y)$ in $F(A_2)$ are the same as the tangent lines to those points in $F(A)$. For all but one $t \in (0,1)$, this implies that the tangent lines to z_0 and $f_A(y)$ are not parallel. From this we conclude that $x_0^* y \neq 0$, otherwise x_0 and y would be distinct eigenvectors of the Hermitian matrix $\operatorname{Re}(e^{-i\theta} A_2)$ for some rotation angle θ, and their tangent lines would be parallel. Therefore the conditions of Lemma 4 apply, and show that g is a selection of f_A^{-1} on all of $F(A)$, except, perhaps, for the line segment from z_0 to the leftmost point in $F(A)$. However, the continuity of g implies that $f_A(g(z)) = z$ must hold for all $z \in F(A)$. □

When the boundary contains a corner point, it is easier to define a continuous selection.

Theorem 4. *Let $A \in M_n(\mathbb{C})$ be a matrix such that $F(A)$ has no weak continuity failures, and there is a corner point $z_0 \in \partial F(A)$. Then there is a continuous selection of f_A^{-1} on $F(A)$.*

Proof. As a corner point, z_0 must be the image of a normal eigenvector x_0 under the action of f_A [6, Theorem 1]. It is also the endpoint of two flat portions of the boundary. Assume the other endpoints are z_1 and z_{k-1} respectively. Since there are no weak continuity failures, there is a continuous selection of the arc of the boundary from z_1 to z_{k-1} opposite z_0. Let $x(t)$ denote the vectors in this selection, with $t \in [1, k-1]$, so that $z(t) = f_A(x(t))$ parametrizes the arc of the boundary, and $z(1) = z_1$ and $z(k-1) = z_{k-1}$. Since x_0 is a normal eigenvector, the compression of A onto the subspace $\operatorname{Span}\{x_0, x(t)\}$ must always be a normal matrix, with $x_0 \perp x(t)$ for all t. Note that every $z \in F(A)\backslash\{z_0\}$ can be written uniquely as

$$z = \lambda z_0 + (1-\lambda)z(t),$$

where $\lambda \in [0, 1)$ and $t \in [1, k-1]$ are continuous functions of z. Let

$$g(z) = \sqrt{\lambda}\, x_0 + \sqrt{1-\lambda}\, x(t).$$

By construction, g is continuous on $F(A)\backslash\{z_0\}$, and if we define $g(z_0) = x_0$, then g extends continuously to all of $F(A)$. Since x_0 is a normal eigenvector of A and is orthogonal to $x(t)$, it follows that $f_A(g(z)) = z$ as desired, so g is a continuous selection of f_A^{-1} on $F(A)$. □

Note that Theorem 4 covers the case when A is any normal matrix.

5. Selections with Weak Continuity Failures

Theorem 5. *Suppose that $A \in M_n(\mathbb{C})$ and f_A^{-1} is weakly continuous on $F(A)$ except at the points $w_1, \ldots, w_k \in \partial F(A)$. For any open set G containing $\{w_1, \ldots, w_k\}$ there is a continuous selection of f_A^{-1} on $F(A)\backslash G$.*

Proof. We will separate the proof into two cases. In the first case, suppose that $F(A)$ has no corner points. Rotate $F(A)$ so that there are no vertical flat portions and no exceptional point on the boundary has a vertical tangent. Let z_0 denote the rightmost point in $F(A)$. As in the proof of Theorem 3, we may construct a path $y : [0, 1] \to \mathbb{C}S^n$ such that f_A is a bijection from $y([0, 1))$ to $\partial F(A)$, $y(0) \in f_A^{-1}(z_0)$, $y(1)$ is a scalar multiple of $y(0)$, and $y(0)^*y(t) \in \mathbb{R}$ for all $t \in (0, 1)$. Unfortunately, $y(t)$ cannot be continuous at points corresponding to weak-continuity failures of f_A^{-1}, but we may construct $y(t)$ so that it is continuous everywhere else.

Choose an ϵ-neighborhood around each w_j such that each neighborhood is contained in G and each neighborhood only contains one exceptional point of $\partial F(A)$, namely the corresponding w_j. In each neighborhood, chose w_j^+ and $w_j^- \in \partial F(A)$ on either side of w_j. There exist t_j^+ and t_j^- such that $w_j^\pm = f_A(y(t_j^\pm))$. We will replace $y(t)$ on the interval $[t_j^-, t_j^+]$ with an alternative path that is continuous in $\mathbb{C}S^n$.

As in the proof of Theorem 3, it will be convenient to let $x_0 = y(0)$. Let $u_j^\pm = y(t_j^\pm)$, and consider the 3-by-3 compression $A_3^{(j)}$ of A corresponding to $\operatorname{Span}\{x_0, u_j^+, u_j^-\}$. By construction $F(A_3^{(j)}) \subset F(A)$. Since $A_3^{(j)}$ is only 3-by-3, the map $f^{-1}_{A_3^{(j)}}$ has no weak continuity failures on $F(A_3)$ [4, Theorem 11]. For each $z \in F(A_3^{(j)})$, the map $f^{-1}_{A_3^{(j)}}(z) \subset f_A^{-1}(z)$, since $f^{-1}_{A_3^{(j)}}$ takes values in $\operatorname{Span}\{x_0, u_j^+, u_j^-\} \cap \mathbb{C}S^n$. Thus, by finding a continuous selection of each $f^{-1}_{A_3^{(j)}}$ on the convex hull $\operatorname{Conv}\{z_0, w_j^+, w_j^-\}$ that agrees with the construction of a continuous selection given

in the proof of Theorem 3 on the line segments from z_0 to w_j^+ and w_j^-, we will find a continuous selection of f_A^{-1} that applies to all of $F(A)$ except for the ϵ-neighborhoods around the weak continuity failure points w_j. In particular, the selection is continuous on $F(A)\backslash G$.

It is necessary to ensure that the map $f^{-1}_{A_3^{(j)}}$ is strongly continuous along the boundary of $F(A_3^{(j)})$ from w_j^- to w_j^+ opposite x_0. By [4, Theorem 11], strong continuity holds at all points in $F(A_3^{(j)})$, except possibly one exceptional point. This will only be a problem if that one point happens to be either w_j^+ or w_j^-, since in that case it may not be possible to find a continuous path $\gamma_j : [t_j^-, t_j^+] \to \mathbb{C}S^n$ such that $\gamma_j(t_j^\pm) = y(t_j^\pm)$, and such that $f_A(\gamma_j(t))$ parametrizes the arc of the boundary of $F(A_3^{(j)})$ between w_j^- and w_j^+.

As shown in the proof of [4, Theorem 11], strong continuity fails for one point on the boundary of $F(A_3^{(j)})$ if and only if $F(A_3^{(j)})$ is a non-degenerate convex ellipse and $A_3^{(j)}$ has a normal eigenvalue on the boundary. As mentioned previously, having a normal eigenvalue on the boundary of $F(A_3^{(j)})$ would not prevent finding a continuous path $\gamma_j(t)$, unless the eigenvalue is either $w_j^\pm$.

Suppose without loss of generality that this is the case, and that w_j^+ is a normal eigenvalue of $A_3^{(j)}$ located on the boundary of the elliptical range $F(A_3^{(j)})$. In this case the 2-by-2 compression of A onto Span $\{x_0, u_j^-\}$ is the same ellipse as $F(A_3^{(j)})$. If the arc of the boundary of $F(A)$ from w_j^+ to w_j coincides at infinitely many points with an elliptical arc of $\partial F(A_3^{(j)})$, then the two critical curves are identical analytic curves. In that case, the arc of the boundary of $F(A)$ from w_j^- to w_j must be a different critical curve. By choosing a different w_j^-, we may ensure that the boundary of $F(A_3^{(j)})$ does not coincide with the arc of the boundary of $F(A)$ from w_j^+ to w_j. Then by choosing a w_j^+ closer to w_j if necessary, we may ensure that both $f^{-1}_{A_3^{(j)}}(w_j^\pm)$ are rank 1, and therefore strong continuity of $f^{-1}_{A_3^{(j)}}$ holds at both $w_j^\pm$. We may now construct a continuous selection of $f^{-1}_{A_3^{(j)}}$ on the arc of the boundary of $F(A_3^{(j)})$ from w_j^- to w_j^+ using (2) or (3) for round portions, and (4) for flat portions. By change of variables, we may assume that the curve γ_j we obtain has domain $[t_j^-, t_j^+]$ and by construction $\gamma_j(t_j^\pm) = u_j^\pm$.

Repeating the argument from the proof of Theorem 3, we use (8) with y replaced by γ_j to define the continuous selection of $f^{-1}_{A_3^{(j)}}$ and therefore f_A^{-1} on $\text{Conv}\{z_0, w_j^-, w_j^+\}$. Note that we do not need to worry about $\gamma_j(t)$ having vertical tangent lines since the slopes of those tangent lines will be arbitrarily close to the slope of the tangent line to w_j in $F(A)$. Once we have a continous selection of f_A^{-1} on each $\text{Conv}\{z_0, w_j^-, w_j^+\}$, the method of Theorem 3 extends those selections to a continuous selection of f_A^{-1} on $F(A)\backslash G$.

In the case where $F(A)$ has a corner point, the argument above can be simplified since we no longer need to worry if the points $w_j^\pm$ are strong continuity failure points of $f^{-1}_{A_3^{(j)}}$. We simply apply the technique of the proof of Theorem 4 directly to $F(A_3^{(j)})$ to obtain a continuous selection of $f^{-1}_{A_3^{(j)}}$ on $\text{Conv}\{z_0, w_j^-, w_j^+\}$ which

extends via the method of Theorem 4 to a continuous selection of f_A^{-1} on all of $F(A)\backslash G$. $\square$

Remark. One might ask whether a continuous selection of f_A^{-1} can be defined on $F(A)\backslash\{w_1, \ldots, w_k\}$, that is, everywhere except the points where weak continuity fails. This is currently an open question.

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